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Programme

1 Lecture 1: Introduction to *Flavour Physics*
2 Lecture 1: (Very brief) Introduction to lattice computations in Flavour Physics
3 Lecture 2: Tree-Level Processes
4 Lecture 2: FCNC Processes
5 Lecture 2: New directions in Lattice Flavour Physics
3. Tree Level Processes - Determination of $V_{us}$: $K_{\ell 2}$ Decays

All QCD effects are contained in a single constant, $f_K$, the kaon’s (leptonic) decay constant.

$$\langle 0 | \bar{s} \gamma^\mu \gamma^5 u | K(p) \rangle = i f_K p^\mu . \quad (f_\pi \simeq 132 \text{MeV})$$

$$\frac{\Gamma(K \to \mu \bar{\nu}(\gamma))}{\Gamma(\pi \to \mu \bar{\nu}(\gamma))} = \frac{|V_{us}|^2}{|V_{ud}|^2} \frac{f_K^2}{f_\pi^2} \frac{m_K}{m_\pi} \left(1 - \frac{m_\mu^2}{m_K^2}\right) \times 0.9930(35)$$

From the experimental ratio of the widths we get:

$$\frac{|V_{us}|}{|V_{ud}|} \frac{f_K}{f_\pi} = 0.2760(4) , \quad \text{M.Moulson, arXiv:1411.5252, J.Rosner, S.Stone & R.Van de Water, arXiv:1509.02220}$$

so that a precise determination of $f_K/f_\pi$ will yield $V_{us}/V_{ud}$.

Every collaboration calculates $f_K$ and $f_\pi$.  

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Hsinchu, 18th May 2017
Determination of $V_{us}$ - $K\ell_3$ Decays

\[ \langle \pi(p_{\pi})|\bar{s}\gamma_\mu u|K(p_K)\rangle = f_0(q^2) \frac{M_K^2 - M_\pi^2}{q^2} q_\mu + f_+(q^2) \left[ (p_{\pi} + p_K)_\mu - \frac{M_K^2 - M_\pi^2}{q^2} q_\mu \right] \]

where $q \equiv p_K - p_{\pi}$.

\[ \Gamma_{K\to\pi\ell\nu} = C_K^2 \frac{G_F^2 m_K^5}{192\pi^3} I_{\text{EW}}[1 + 2\Delta_{\text{SU}(2)} + \Delta_{\text{EM}}] |V_{us}|^2 |f_+(0)|^2 \]

From the experimental measurement of the width we get:

\[ |V_{us}|f_+(0) = 0.2165(4) , \quad \text{M.Moulson, arXiv:1411.5252} \]

so that a precise determination of $f_+(0)$ will yield $V_{us}$. 
We have the two precise results:

\[
\frac{|V_{us} f_K|}{V_{ud} f_\pi} = 0.2760(4) \quad \text{and} \quad |V_{us} f_+(0)| = 0.2165(4)
\]

We can view these as two equations for the four unknowns \( f_K / f_\pi, f_+(0), V_{us} \) and \( V_{ud} \).

Within the Standard Model we also have the unitarity constraint:

\[
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1
\]

(Alternatively the current value of \( V_{ub} = 4.13(49) \times 10^{-3} \) can be inserted, changing the value of \( V_{us} \) in the fourth decimal place.)

Thus we now have 3 equations for four unknowns.

There has been considerable work recently in updating the determination of \( V_{ud} \) based on 20 different superallowed transitions.\(^\text{1}\)\(^\text{1}\)

\[
|V_{ud}| = 0.97417(21).
\]

If we accept this value then we are able to determine the remaining 3 unknowns:

\[
|V_{us}| = 0.2258(9), \quad f_+(0) = 0.9587(42), \quad \frac{f_K}{f_\pi} = 1.1907(51).
\]
$V_{us}$ from Lattice Simulations

Flavianet Lattice Averaging Group - arXiv:1607.00299
Unitarity and the First Row of the CKM Matrix

Lattice results are $2\sigma$ consistent with the unitarity of the CKM Matrix

- For $N_f = 2 + 1$ simulations FLAG quotes the following current values:
  
  \[ \frac{f_K}{f_\pi} = 1.192(5) \quad \text{and} \quad f_+(0) = 0.9677(23)(33). \]

- Taking the experimental results for $K_{\ell2}$ and $K_{\ell3}$ decays and dividing by the $N_f = 2 + 1$ lattice values of $f_K/f_\pi$ and $f_+(0)$ gives:
  
  \[ V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.984(11). \]

- If we combine the experimental results with the value of $V_{ud}$ and the lattice values of $f_+(0)$ or $f_K/f_\pi$ we find:
  
  \[ V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.9991(6) \quad \text{or} \quad V_{ud}^2 + V_{us}^2 + V_{ub}^2 = 0.9999(6). \]

- Very significant test of universality of coupling of "W"-like bosons to quarks and leptons.

- Private question: At such level of precision, are there still continuum (and perhaps chiral) effects to be controlled fully?
In her review talk at Lattice 2016, *Impact of Lattice QCD on CKM Phenomenology*, Monika Blanke quoted the PDG value

\[ V_{us} = 0.2248(6) \]

and we have seen how \( V_{us} \) is determined. 

In a similar way (but with a special treatment of the heavy quarks one can obtain \( V_{cb} \) from semileptonic decays:

\[ |V_{cb}|_{B \rightarrow D^* \ell \nu} = (39.04 \pm 0.75) \times 10^{-3} \]

and

\[ |V_{cb}|_{B \rightarrow D \ell \nu} = (40.49 \pm 0.97) \times 10^{-3} \]

There is a tension between these values and the inclusive one (to whose determination lattice simulations do not contribute)

\[ |V_{cb}|_{incl} = (42 \pm 0.65) \times 10^{-3} \]

The uncertainty in the determination of \( V_{cb} \) restricts the precision of much flavour phenomenology.
There is an even larger discrepancy between the inclusive and exclusive values for $V_{ub}$ (for references to experimental and lattice papers see Blanke’s review):

$$|V_{ub}|_{B \to \pi \ell \nu} = (3.72 \pm 0.16) \times 10^{-3} \quad \text{and} \quad |V_{ub}|_{\text{incl}} = (4.41 \pm 0.15^{+0.15}_{-0.19}) \times 10^{-3}.$$
Finally, let me mention that Brod and Zupan showed that the angle $\gamma$ is determined to an excellent precision from tree-level $B \rightarrow DK$ decays

and the LHCb collaboration has used such techniques to obtain

$$\gamma = \left(72.2^{+6.8}_{-7.3}\right)^\circ$$

This result is largely statistics limited and will be improved very significantly in the future.

In spite of the very significant progress there is scope and need to improve the determination of $V_{us}$, $V_{cb}$ and $V_{ub}$ further.

For $B$-decays it is very important that more collaborations become involved and this is happening.
4. FCNC Processes - $\varepsilon_K$ and Neutral Kaon Mixing

- In the evaluation of $\varepsilon_K$, we need to calculate (schematically)

$$K^0 \rightarrow \bar{K}^0$$

(gluons and quark loops not shown.)

- The process is short-distance dominated and so we can approximate the above by a perturbatively calculable (Wilson) coefficient $C$ times

$$K^0 \rightarrow \bar{K}^0$$

where the black dot represents the insertion of the local operator $(\bar{s}\gamma_\mu (1 - \gamma^5)d)(\bar{s}\gamma_\mu (1 - \gamma^5)d)$.

- In the standard model only this single operator contributes.
- In generic BSM theories there are 5 possible $\Delta S = 2$ operators contributing.
FCNC Processes - $\varepsilon_K$ and Neutral Kaon Mixing

$$
\varepsilon_K = \frac{A(K_L \rightarrow (\pi\pi)_{I=0})}{A(K_S \rightarrow (\pi\pi)_{I=0})} = e^{i\phi_\varepsilon} \sin\phi_\varepsilon \left[ \frac{\text{Im}\langle \bar{K}^0 | H^{\Delta S=2}_W | K^0 \rangle}{\Delta m_K} \right] + \text{L.D. effects}
$$

where

$$
|\varepsilon_K| = 2.228(11) \times 10^{-3}
$$

$$
\phi_\varepsilon = \arctan \frac{\Delta m_K}{\Delta \Gamma_K / 2} = 43.52(5)^\circ
$$

$$
\Delta m_K = m_{K_L} - m_{K_S} = 3.4839(59) \times 10^{-12} \text{ MeV}
$$

$$
\Delta \Gamma_K = \Gamma_S - \Gamma_L = 7.3382(33) \times 10^{-15} \text{ GeV}
$$

- It is conventional to present the short-distance contribution in terms of the $B_K$ parameter:

$$
\langle \bar{K}^0 | H^{\Delta S=2}_W | K^0 \rangle \propto \langle \bar{K}^0 | (\bar{s} \gamma^\mu (1 - \gamma^5) d) (\bar{s} \gamma_\mu (1 - \gamma^5) d) | K^0 \rangle \equiv \frac{8}{3} f_K^2 m_K^2 B_K(\mu).
$$

- Lattice calculations of $B_K$ have been performed since the mid 1980s. The precision is now such that the $O(5\%)$ long-distance (LD) effects have to be considered.

Buras, Guadagnoli, arXiv:0805.3887

Buras, Guadagnoli, Isidori arXiv:1002.3612
Results for $B_K$ (FLAG)

- FLAG-3 quote from simulations with $N_f = 2 + 1$:
  \[ \hat{B}_K = 0.7625(97) \simeq 0.76(1). \]

- The FLAG-1 result was $\hat{B}_K = 0.738(20)$ and at EPS 1993 I quoted a summary $\hat{B}_K = 0.8(2)$. M. Lusignoli, L. Maiani, G. Martinelli and L. Reina, Nucl.Phys. B369 (1992) 139

- The dominant contribution to $\varepsilon_K \propto |V_{cb}|^4$ and PDG(2016) quote $|V_{cb}| = (41.1 \pm 1.3) \times 10^{-3}$ error on $B_K$ is no longer the dominant one.

- Among our (RBC-UKQCD) main projects are the evaluation of $\Delta M_K$ and the long-distance contributions to $\varepsilon_K$. 

Chris Sachrajda
Hsinchu, 18th May 2017
Beyond the standard model there are in general 5 independent operators which contribute neutral Kaon mixing:

\[ \mathcal{H}^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu) Q_i^{\Delta S=2}(\mu). \]

The five operators are:

\[ Q_1^{\Delta S=2} = [\bar{s}^i \gamma_{\mu} (1 - \gamma_5) d^i ] [\bar{s}^j \gamma_{\mu} (1 - \gamma_5) d^j ] \]
\[ Q_2^{\Delta S=2} = [\bar{s}^i (1 - \gamma_5) d^i ] [\bar{s}^j (1 - \gamma_5) d^j ] \]
\[ Q_3^{\Delta S=2} = [\bar{s}^i (1 - \gamma_5) d^i ] [\bar{s}^j (1 - \gamma_5) d^j ] \]
\[ Q_4^{\Delta S=2} = [\bar{s}^i (1 - \gamma_5) d^i ] [\bar{s}^j (1 + \gamma_5) d^j ] \]
\[ Q_5^{\Delta S=2} = [\bar{s}^i (1 - \gamma_5) d^i ] [\bar{s}^j (1 + \gamma_5) d^j ] \]

\( i,j \) are colour indices.

The matrix elements can be calculated in a similar way to \( B_K \). For a recent study and references to the original literature see Boyle, Garron and Hudspith.

\[ \text{arXiv:1206.5737} \]

\( Q_1^{\Delta S=2} \) transforms as (27,1) under \( SU(3)_L \times SU(3)_R \), \( Q_2^{\Delta S=2} \) and \( Q_3^{\Delta S=2} \) as (6, \overline{6}) and \( Q_4^{\Delta S=2} \) and \( Q_5^{\Delta S=2} \) as (8,8) \( \Rightarrow \) Renormalization matrix is block diagonal.
“Standard” lattice calculations in flavour physics are of matrix elements of local operators between single hadron states \( \langle h_2(p_2) | O(0) | h_1(p_1) \rangle \) (or \( \langle 0 | O(0) | h(p) \rangle \)).

For the remainder of this lecture I will present some selected recent developments which go beyond the standard calculations.

1. \( K \rightarrow \pi \pi \) decays;
2. Long-distance contributions to physical processes, illustrated by rare kaon decays \( K \rightarrow \pi \ell^+ \ell^- \) and \( K \rightarrow \pi \nu \bar{\nu} \) decays;
3. Isospin corrections (including electromagnetic effects).
5.1 Direct Evaluation of $K \rightarrow \pi\pi$ Decays

- $K \rightarrow \pi\pi$ decays are a very important class of processes for standard model phenomenology.

- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.

- Among the interesting issues are the origin of the $\Delta I = 1/2$ rule ($\text{Re}A_0/\text{Re}A_2 \simeq 22.5$) and an understanding of the experimental value of $\varepsilon'/\varepsilon$, the parameter which was the first experimental evidence of direct CP-violation.

- The evaluation of $K \rightarrow \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0 \mid O(0) \mid h \rangle$ and $\langle h_2 \mid O(0) \mid h_1 \rangle$ matrix elements with a single hadron in the initial and/or final state.
Directly $CP$-violating decays are those in which a $CP$-even (-odd) state decays into a $CP$-odd (-even) one: 

$$K_L \propto K_2 + \bar{\epsilon}K_1.$$ 

Consider the following contributions to $K \rightarrow \pi\pi$ decays:

- **Direct ($\epsilon'$)**
  - $I = 0$, Complex
  - $I = 0$, Real
  - $I = 0$ or 2, Real

Direct $CP$-violation in kaon decays manifests itself as a non-zero relative phase between the $I = 0$ and $I = 2$ amplitudes.

We also have *strong phases*, $\delta_0$ and $\delta_2$, which are independent of the form of the weak Hamiltonian.
Effective Hamiltonian for $K \to \pi\pi$ Decays

\[ \mathcal{H}_{\text{eff}}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} \ V_{ud} V_{us}^* \sum_{i=1}^{10} \left[ z_i(\mu) + \tau y_i(\mu) \right] Q_i, \text{ where } \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \]

Current – Current Operators

\[ Q_1 = (\bar{s}d)_L (\bar{u}u)_L \quad Q_2 = (\bar{s}^i d^i)_L (\bar{u}^i u^i)_L \]

QCD Penguin Operators

\[ Q_3 = (\bar{s}d)_L \sum_{q=u,d,s} (\bar{q}q)_L \quad Q_4 = (\bar{s}^i d^i)_L \sum_{q=u,d,s} (\bar{q}^i q^i)_L \]
\[ Q_5 = (\bar{s}d)_L \sum_{q=u,d,s} (\bar{q}q)_R \quad Q_6 = (\bar{s}^i d^i)_L \sum_{q=u,d,s} (\bar{q}^i q^i)_R \]

Electroweak Penguin Operators

\[ Q_7 = \frac{3}{2} (\bar{s}d)_L \sum_{q=u,d,s} e_q (\bar{q}q)_L \quad Q_8 = \frac{3}{2} (\bar{s}^i d^i)_L \sum_{q=u,d,s} e_q (\bar{q}^i q^i)_L \]
\[ Q_9 = \frac{3}{2} (\bar{s}d)_L \sum_{q=u,d,s} e_q (\bar{q}q)_R \quad Q_{10} = \frac{3}{2} (\bar{s}^i d^i)_L \sum_{q=u,d,s} e_q (\bar{q}^i q^i)_R \]

This 10 operator basis is very natural but over-complete:

\[ Q_{10} - Q_9 = Q_4 - Q_3 \]
\[ Q_4 - Q_3 = Q_2 - Q_1 \]
\[ 2Q_9 = 3Q_1 - Q_3. \]
In 2015 RBC-UKQCD published our first result for $\varepsilon'/\varepsilon$ computed at physical quark masses and kinematics, albeit still with large errors:

$$\left. \frac{\varepsilon'}{\varepsilon} \right|_{\text{RBC-UKQCD}} = (1.38 \pm 5.15 \pm 4.59) \times 10^{-4}$$

to be compared with

$$\left. \frac{\varepsilon'}{\varepsilon} \right|_{\text{Exp}} = (16.6 \pm 2.3) \times 10^{-4}.$$ 

This is by far the most complicated project that I have ever been involved with.

This single result hides much important (and much more precise) information which we have determined along the way.

In this section I will review the main obstacles to computing $K \rightarrow \pi\pi$ decay amplitudes, the techniques used to overcome them and our main results.
$A_0$ and $A_2$ amplitudes with unphysical quark masses and with the pions at rest.

"$K$ to $\pi\pi$ decay amplitudes from lattice QCD,"


"Kaon to two pions decay from lattice QCD, $\Delta I = 1/2$ rule and CP violation"


$A_2$ at physical kinematics and a single coarse lattice spacing.

"The $K \to (\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD,"


"Lattice determination of the $K \to (\pi\pi)_{I=2}$ Decay Amplitude $A_2$"


"Emerging understanding of the $\Delta I = 1/2$ Rule from Lattice QCD,"


Status of RBC-UKQCD calculations of $K \rightarrow \pi \pi$ decays (Cont.)

3. $A_2$ at physical kinematics on two finer lattices $\Rightarrow$ continuum limit taken.

“$K \rightarrow \pi \pi \Delta I = 3/2$ decay amplitude in the continuum limit,”


4. $A_0$ at physical kinematics and a single coarse lattice spacing.

“Standard-model prediction for direct CP violation in $K \rightarrow \pi \pi$ decay,”


To be continued!
The Maiani-Testa Theorem

\[ t_H \pi, \vec{p}_\pi = \vec{q} \]
\[ t_K \pi, \vec{p}_K = 0 \]

- \( K \to \pi\pi \) correlation function is dominated by lightest state, i.e. the state with two-pions at rest. 

Maiani and Testa, PL B245 (1990) 585

\[ C(t_\pi) = A + B_1 e^{-2m_\pi t_\pi} + B_2 e^{-2E_\pi t_\pi} + \ldots \]

Solution 1: Study an excited state. 

Lellouch and Lüscher, hep-lat/0003023

Solution 2: Introduce suitable boundary conditions such that the \( \pi\pi \) ground state is \(|\pi(\vec{q})\pi(-\vec{q})\rangle\).

RBC-UKQCD, C.h.Kim hep-lat/0311003

For \( B \)-decays, with so many intermediate states below threshold, this is the main obstacle to producing reliable calculations.
For $A_2$, there is no vacuum subtraction and we can use the Wigner-Eckart theorem to write

$$
\langle \pi\pi \rangle_{I=2}^I \mid Q_{\Delta I=3/2}^{\Delta I=3/2} \mid K^+ \rangle = \frac{3}{2} \langle \pi\pi \rangle_{I=2}^I \mid Q_{\Delta I=3/2}^{\Delta I=3/2} \mid K^+ \rangle,
$$

and impose anti-periodic conditions on the $d$-quark in one or more directions.

If we impose the anti-periodic boundary conditions in all 3 directions then the ground state is

$$
\left| \pi \left( \frac{\pi}{L}, \frac{\pi}{L}, \frac{\pi}{L} \right) \pi \left( -\frac{\pi}{L}, -\frac{\pi}{L}, -\frac{\pi}{L} \right) \rightangle.
$$

With an appropriate choice of $L$ and the number of directions, we can arrange that $E_{\pi\pi} = m_K$.

Isospin breaking by the boundary conditions is harmless here.
These are based on the Poisson summation formula:

\[
\frac{1}{L} \sum_{n=-\infty}^{\infty} f(p_n^2) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2) + \sum_{n \neq 0} \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2) e^{ipnL},
\]

For single-hadron states the finite-volume corrections decrease exponentially with the volume \(\propto e^{-m_\pi L}\). For multi-hadron states, the finite-volume corrections generally fall as powers of the volume.

For two-hadron states, there is a huge literature following the seminal work by Lüscher and the effects are generally understood.

- The \(K \rightarrow \pi\pi\) amplitudes are obtained from the finite-volume matrix elements by the Lellouch-Lüscher factor which contains the derivative of the phase-shift.
  L.Lellouch & M.Lüscher, hep-lat/0003023,
  C.h.Kim, CTS & S.R.Sharpe, hep-lat/0507006 · · ·

- Recently we have also determined the finite-volume corrections for \(\Delta m_K = m_{KL} - m_{KS}\).

For three-hadron states, there has been a major effort by Hansen and Sharpe leading to much theoretical clarification.

Error budgets in our calculation of $A_2$

<table>
<thead>
<tr>
<th>Source</th>
<th>Re$A_2$</th>
<th>Im$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NPR (nonperturbative)</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
<tr>
<td>NPR (perturbative)</td>
<td>2.9%</td>
<td>7.0%</td>
</tr>
<tr>
<td>Finite volume corrections</td>
<td>2.4%</td>
<td>2.6%</td>
</tr>
<tr>
<td>Unphysical kinematics</td>
<td>4.5%</td>
<td>1.1%</td>
</tr>
<tr>
<td>Wilson coefficients</td>
<td>6.8%</td>
<td>10%</td>
</tr>
<tr>
<td>Derivative of the phase shift</td>
<td>1.1%</td>
<td>1.1%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>9%</strong></td>
<td><strong>12%</strong></td>
</tr>
</tbody>
</table>

- *Wilson Coefficients* and NPR(perturbative) errors are not from our lattice calculation.
- Step-scaling can be used to increase the scale at which the matching is performed.

RBC-UKQCD, T.Blum et al., arXiv:1502:00263
Our first results for $A_2$ at physical kinematics were obtained at a single, rather coarse, value of the lattice spacing ($a \simeq 0.14$ fm). Estimated discretization errors at 15%.

Our recent results were obtained on two new ensembles, $48^3$ with $a \simeq 0.11$ fm and $64^3$ with $a \simeq 0.084$ fm so that we can make a continuum extrapolation:

$$\text{Re}(A_2) = 1.50(4)_{\text{stat}}(14)_{\text{syst}} \times 10^{-8} \text{ GeV.}$$

$$\text{Im}(A_2) = -6.99(20)_{\text{stat}}(84)_{\text{syst}} \times 10^{-13} \text{ GeV.}$$

The experimentally measured value is $\text{Re}(A_2) = 1.479(4) \times 10^{-8}$ GeV.

Although the precision can still be significantly improved (partly by perturbative calculations), the calculation of $A_2$ at physical kinematics can now be considered as standard.
Re $A_2$ is dominated by a simple operator:

$$O^{3/2}_{(27,1)} = (\bar{s}^i d^i)_L \left\{ (\bar{u}^j u^j)_L - (\bar{d}^j d^j)_L \right\} + (\bar{s}^i u^i)_L (\bar{u}^j d^j)_L$$

and two diagrams:

- Re $A_2$ is proportional to $C_1 + C_2$.
- The contribution to Re $A_0$ from $Q_2$ is proportional to $2C_1 - C_2$ and that from $Q_1$ is proportional to $C_1 - 2C_2$ with the same overall sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
- We find instead that $C_2 \approx -C_1$ so that $A_2$ is significantly suppressed!
- We believe that the strong suppression of Re $A_2$ and the (less-strong) enhancement of Re $A_0$ is a major factor in the $\Delta I = 1/2$ rule.
Evidence for the Suppression of $\text{Re} A_2$

Physical Kinematics

- Notation $i \equiv C_i$, $i = 1, 2$.

- Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we needed to compute $\text{Re} A_0$ at physical kinematics and reproduce the experimental value of 22.5.

- Much early phenomenology was based on the vacuum insertion approach. although the qualitative picture we find had been suggested by Bardeen, Buras and Gerard in 1987.

\[ m_\pi \simeq 330 \text{ MeV at threshold.} \]
Calculation of $A_0$

- The calculation is much more difficult for the $K \to (\pi\pi)_{I=0}$ amplitude $A_0$:
  - The presence of disconnected diagrams, vacuum subtraction, ultra-violet power divergences, …
  - $|\pi^+ (\pi/L)\pi^- (-\pi/L)\rangle$ has a different energy from $|\pi^0 (\bar{0})\pi^0 (\bar{0})\rangle$.
  - We have developed the implementation of $G$-parity boundary conditions in which $(u,d) \to (\bar{d}, -\bar{u})$ at the boundary.

Computations were performed on a $32^3 \times 64$ lattice with the Iwasaki and DSDR gauge action and $N_f = 2 + 1$ flavours of Möbius Domain Wall Fermions:

$$a^{-1} = 1.379(7) \text{GeV}, \ m_\pi = 143.2(2.0) \text{MeV}, (E_\pi = 274.8(1.4) \text{MeV})$$

The $\pi\pi$ energies are

$$E_{\pi\pi}^{I=0} = (498 \pm 11) \text{MeV} \quad E_{\pi\pi}^{I=2} = (565.7 \pm 1.0) \text{MeV}$$

to be compared with $m_K = (490.6 \pm 2.4) \text{MeV}$.

Lüscher’s quantisation condition $\Rightarrow E_{\pi\pi}^{I=0}$ corresponds to $\delta_0 = (23.8 \pm 4.9 \pm 1.2)^\circ$, which is somewhat smaller than phenomenological expectations.
\[ H_W = \frac{G_F}{\sqrt{2}} V_{us}^* V_{ud} \sum_{i=1}^{10} [z_i(\mu) + \tau y_i(\mu)] Q_i(\mu). \]

\[ \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \]


<table>
<thead>
<tr>
<th>i</th>
<th>Re((A_0))(GeV)</th>
<th>Im((A_0))(GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.02(0.20)(0.07) \times 10^{-7}</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3.60(0.90)(0.28) \times 10^{-7}</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1.28(1.69)(1.20) \times 10^{-10}</td>
<td>1.53(2.03)(1.44) \times 10^{-12}</td>
</tr>
<tr>
<td>4</td>
<td>-2.01(0.69)(0.36) \times 10^{-9}</td>
<td>1.80(0.61)(0.32) \times 10^{-11}</td>
</tr>
<tr>
<td>5</td>
<td>-8.93(2.23)(1.84) \times 10^{-10}</td>
<td>1.54(0.38)(0.32) \times 10^{-12}</td>
</tr>
<tr>
<td>6</td>
<td>3.51(0.89)(0.23) \times 10^{-9}</td>
<td>-3.56(0.90)(0.24) \times 10^{-11}</td>
</tr>
<tr>
<td>7</td>
<td>2.38(0.40)(0.00) \times 10^{-11}</td>
<td>8.49(1.44)(0.00) \times 10^{-14}</td>
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<tr>
<td>8</td>
<td>-1.28(0.04)(0.00) \times 10^{-10}</td>
<td>-1.71(0.05)(0.00) \times 10^{-12}</td>
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<td>9</td>
<td>-7.38(1.97)(0.48) \times 10^{-12}</td>
<td>-2.41(0.64)(0.16) \times 10^{-12}</td>
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<tr>
<td>10</td>
<td>7.29(2.62)(0.68) \times 10^{-12}</td>
<td>-4.72(1.69)(0.44) \times 10^{-13}</td>
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Total (stat only) 4.66(0.96)(0.27) \times 10^{-7} -1.90(1.19)(0.32) \times 10^{-11}

Final (incl. syst) 4.66(1.00)(1.21) \times 10^{-7} -1.90(1.23)(1.04) \times 10^{-11}
Representative Errors

<table>
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<th>Description</th>
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<td>Finite lattice spacing</td>
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<td>Wilson coefficients</td>
<td>12%</td>
<td>Excited states</td>
<td>≤ 5%</td>
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<td>Parametric errors</td>
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<td>Operator renormalization</td>
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<td>≤ 3%</td>
<td>Lellouch-Lüscher factor</td>
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</tr>
<tr>
<td>Total (added in quadrature)</td>
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<td>26%</td>
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Two groups have improved the renormalisation group running and used our matrix elements to obtain $\varepsilon'/\varepsilon$:

- **RBC-UKQCD**
  - A.Buras, M.Gorbahn, S.Jäger & M.Jamin
  - $1.38 \pm 5.15 \pm 4.59 \times 10^{-4}$
  - arXiv:1505.07863

  $1.9 \pm 4.5 \times 10^{-4}$
  - arXiv:1507.06345

- **T.Kitahara, U.Nierste & P.Tremper**
  - $(1.06 \pm 5.07) \times 10^{-4}$
  - arXiv:1607.06727

and recall the experimental value is $(16.6 \pm 2.3) \times 10^{-4}$. 
Conclusions for $K \to \pi\pi$ decays

- As a result of our work, the computation of $A_2$ is now "standard".
- It appears that the explanation of the $\Delta I = 1/2$ rule has a number of components, of which the significant cancelation between the two dominant contributions to $\text{Re}A_2$ is a major one.
- We have completed the first calculation of $\varepsilon'/\varepsilon$ with controlled errors $\Rightarrow$ motivation for further refinement (systematic improvement by collecting more statistics, working on larger volumes, $\geq 2$ lattice spacings etc.)
- $\varepsilon'/\varepsilon$ is now a quantity which is amenable to lattice computations.
5.2 Long-distance contributions to rare kaon decays

- The mission of lattice calculations is to evaluate hadronic effects.
- "Standard" lattice calculations in flavour physics are of matrix elements of local operators between single hadron states $\langle h_2(p_2)|O(0)|h_1(p_1)\rangle$ (or $\langle 0|O(0)|h(p)\rangle$).
- For example, in the evaluation of $\varepsilon_K$, we need to calculate (schematically)

  \[
  K^0 \rightarrow \bar{K}^0 \quad (\text{gluons and quark loops not shown.})
  \]

- The process is short-distance dominated and so we can approximate the above by a perturbatively calculable (Wilson) coefficient $C$ times

  \[
  \begin{align*}
  &K^0 \rightarrow \bar{s} \rightarrow \bar{d} \rightarrow \bar{K}^0 \\
  &\quad d \rightarrow s
  \end{align*}
  \]

  where the black dot represents the insertion of the local operator $(\bar{s}\gamma_\mu (1 - \gamma^5)d)\ (\bar{s}\gamma_\mu (1 - \gamma^5)d)$.

  - In the standard model only this single operator contributes.
  - In generic BSM theories there are 5 possible $\Delta S = 2$ operators contributing.
Long-distance contributions to flavour changing processes are characterised by matrix elements of bilocal (in general, multilocal) operators:

\[ \int \int d^4x \, d^4y \, \langle f | T[Q_1(x)Q_2(y)] | i \rangle. \]

Here this will be illustrated through the important application to rare kaon decays.

1. \( K \rightarrow \pi \ell^+ \ell^- \) decays.
2. \( K^+ \rightarrow \pi^+ \nu \bar{\nu} \) decays.

(Other applications being studied by the RBC-UKQCD Collaboration* include the \( K_L-K_S \) mass difference and \( \varepsilon_K \).)

* RBC=Riken Research Center, Brookhaven National Laboratory, Columbia University; UKQCD = Edinburgh + Southampton.
“Prospects for a lattice computation of rare kaon decay amplitudes: I, $K \to \pi \ell^+ \ell^-$ decays"
N.H.Christ, X.Feng, A.Portelli and CTS

“Prospects for a lattice computation of rare kaon decay amplitudes: II, $K \to \pi \nu \bar{\nu}$ decays"
N.H.Christ, X.Feng, A.Portelli and CTS

“First exploratory calculation of the long distance contributions to the rare kaon decay $K \to \pi \ell^+ \ell^-$"
N.H.Christ, X.Feng, A.Jüttner, A.Lawson, A.Portelli and CTS

“Exploratory lattice QCD study of the rare kaon decay $K^+ \to \pi^+ \nu \bar{\nu}$”
arXiv:1701.02858
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<td>8</td>
<td>“Long distance part of $\varepsilon_K$ from lattice QCD”</td>
<td>Z.Bai</td>
<td>arXiv:1611.06601, (Journal paper in preparation)</td>
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<td>+ a number of conference papers</td>
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Physics Motivation: $K \rightarrow \pi \nu \bar{\nu}$ Decays

- NA62 ($K^+ \rightarrow \pi^+ \nu \bar{\nu}$) and KOTO ($K_L \rightarrow \pi^0 \nu \bar{\nu}$) are beginning their experimental programme to study these decays. These FCNC processes provide ideal probes for the observation of new physics effects.

- The dominant contributions from the top quark $\Rightarrow$ they are also very sensitive to $V_{ts}$ and $V_{td}$.

- Experimental results and bounds:
  \[
  \text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{exp}} = 1.73^{+1.15}_{-1.05} \times 10^{-10} \quad \text{A.Artamonov et al. (E949), arXiv:0808.2459}
  \]
  \[
  \text{Br}(K_L \rightarrow \pi^0 \nu \bar{\nu}) \leq 2.6 \times 10^{-8} \text{ at 90\% confidence level} , \quad \text{J.Ahn et al. (E291a), arXiv:0911.4789}
  \]

- Sample recent theoretical predictions:
  \[
  \text{Br}(K^+ \rightarrow \pi^+ \nu \bar{\nu})_{\text{SM}} = (9.11 \pm 0.72) \times 10^{-11} \quad \text{A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693}
  \]
  \[
  \text{Br}(K_L \rightarrow \pi^0 \nu \bar{\nu})_{\text{SM}} = (3.00 \pm 0.30) \times 10^{-11} , \quad \text{A.Buras, D.Buttazzo, J.Girrbach-Noe, R.Knejgens, arXiv:1503.02693}
  \]

- To what extent can lattice calculations reduce the theoretical uncertainty?
Short and Long-Distance Contributions

- To what extent can lattice calculations reduce the theoretical uncertainty?

- $K \rightarrow \pi \nu \bar{\nu}$ decays are SD dominated and the hadronic effects can be determined from CC semileptonic decays such as $K^+ \rightarrow \pi^0 e^+ \nu$.
  - Lattice calculations of the $K_{\ell 3}$ form factors are well advanced, FLAG review, S.Aoki et al, arXiv:1607.00299

- LD contributions, i.e. contributions from distances greater than $1/m_c$ are negligible for $K_L$ decays and are expected to be $O(5\%)$ for $K^+$ decays.
  - $K_L$ decays are therefore one of the cleanest places to search for the effects of new physics.

- The aim of our study is to compute the LD effects in $K^+$ decays. These provide a significant, if probably still subdominant, contribution to the theoretical uncertainty (which is dominated by the uncertainties in CKM matrix elements).

- A phenomenological estimate of the long distance effects, estimated these to enhance the branching fraction by 6% with an uncertainty of 3%.

- Lattice QCD can provide a first-principles determination of the LD contribution with controlled errors.
  - Given the NA62 experiment, it is timely to perform a lattice QCD calculation of these effects.
Physics Motivation: $K \to \pi \ell^+ \ell^-$ Decays

Some comments from F. Mescia, C. Smith, S. Trine hep-ph/0606081

- Rare kaon decays which are dominated by short-distance FCNC processes, $K \to \pi \nu \bar{\nu}$ in particular, provide a potentially valuable window on new physics at high-energy scales.

- The decays $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ are also considered promising because the long-distance effects are reasonably under control using ChPT.

  - They are sensitive to different combinations of short-distance FCNC effects and hence in principle provide additional discrimination to the neutrino modes.
  
  - A challenge for the lattice community is therefore to calculate the long-distance effects reliably.
There are three main contributions to the amplitude:

1. **Short distance contributions:**

   \[ H_{\text{eff}} = - \frac{G_F \alpha}{\sqrt{2}} V_{ts}^* V_{td} \left\{ y_7 V (\bar{s} \gamma_\mu d) (\bar{\ell} \gamma^\mu \ell) + y_7 A (\bar{s} \gamma_\mu d) (\bar{\ell} \gamma^\mu \gamma_5 \ell) \right\} + \text{h.c.} \]

   - Direct CP-violating contribution.
   - In BSM theories other effective interactions are possible.

2. **Long-distance indirect CP-violating contribution**

   \[ A_{ICPV}(K_L \rightarrow \pi^0 \ell^+ \ell^-) = \varepsilon A(K_1 \rightarrow \pi^0 \ell^+ \ell^-) \simeq \varepsilon A(K_S \rightarrow \pi^0 \ell^+ \ell^-). \]

3. **The two-photon CP-conserving contribution** \( K_L \rightarrow \pi^0 (\gamma^* \gamma^* \rightarrow \ell^+ \ell^-) \).

\[ K_L \rightarrow \pi^0 \ell^+ \ell^- \text{ Decays} \]
The current phenomenological status for the SM predictions is nicely summarised by:

\[ \text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = 10^{-12} \times \left\{ 15.7 |a_S|^2 \pm 6.2 |a_S| \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right) + 2.4 \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right)^2 \right\} \]

\[ \text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} = 10^{-12} \times \left\{ 3.7 |a_S|^2 \pm 1.6 |a_S| \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right) + 1.0 \left( \frac{\text{Im} \lambda_t}{10^{-4}} \right)^2 \right\} \]

\[ \lambda_t = V_{td} V_{ts}^* \text{ and } \text{Im} \lambda_t \simeq 1.35 \times 10^{-4}. \]

\[ |a_S|, \text{ the amplitude for } K_S \to \pi^0 \ell^+ \ell^- \text{ at } q^2 = 0 \text{ as defined below, is expected to be } O(1) \text{ but the sign of } a_S \text{ is unknown. } |a_S| = 1.06^{+0.26}_{-0.21}. \]

For \( \ell = e \) the two-photon contribution is negligible.

Taking the positive sign (?) the prediction is

\[ \text{Br}(K_L \to \pi^0 e^+ e^-)_{\text{CPV}} = (3.1 \pm 0.9) \times 10^{-11} \]

\[ \text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPV}} = (1.4 \pm 0.5) \times 10^{-11} \]

\[ \text{Br}(K_L \to \pi^0 \mu^+ \mu^-)_{\text{CPC}} = (5.2 \pm 1.6) \times 10^{-12}. \]

The current experimental limits (KTeV) are:

\[ \text{Br}(K_L \to \pi^0 e^+ e^-) < 2.8 \times 10^{-10} \quad \text{and} \quad \text{Br}(K_L \to \pi^0 \mu^+ \mu^-) < 3.8 \times 10^{-10}. \]
We now turn to the CPC decays $K_S \to \pi^0 \ell^+ \ell^-$ and $K^+ \to \pi^+ \ell^+ \ell^-$ and consider

$$T_i^{\mu} = \int d^4 x e^{-i q \cdot x} \langle \pi(p) | T \{ J_{\text{em}}^\mu(x) Q_i(0) \} | K(k) \rangle,$$

where $Q_i$ is an operator from the $\Delta S = 1$ effective weak Hamiltonian.

EM gauge invariance implies that

$$T_i^{\mu} = \frac{\omega_i(q^2)}{(4\pi)^2} \left\{ q^2 (p+k)^\mu - (m_K^2 - m_\pi^2) q^\mu \right\}.$$

Within ChPT the low energy constants $a_+$ and $a_S$ are defined by

$$a = \frac{1}{\sqrt{2}} V_{us}^* V_{ud} \left\{ C_1 \omega_1(0) + C_2 \omega_2(0) + \frac{2N}{\sin^2 \theta_W} f_+(0) C_{7V} \right\}$$

where $Q_{1,2}$ are the two current-current GIM subtracted operators and the $C_i$ are the Wilson coefficients. ($C_{7V}$ is proportional to $y_{7V}$ above).

Phenomenological values: $a_+ = -0.578 \pm 0.016$ and $|a_S| = 1.06^{+0.26}_{-0.21}$.

What can we achieve in lattice simulations?
Much of the general theoretical framework will be presented in the context of $K \rightarrow \pi \ell^+ \ell^-$ decays. 

Issues specific to $K \rightarrow \pi \nu \bar{\nu}$ decays will be discussed later.
In Euclidean space we calculate correlation functions of the form

\[ C \equiv \int_{-T_a}^{T_b} dt_x \int d^3x \left\langle \phi_{\pi}(\vec{p}, t\pi) T[J(0)H(x)] \phi_{\pi}^+(\vec{p}_K, tK) \right\rangle \equiv \sqrt{Z_K} \frac{e^{-E_K|t_K|}}{2m_K} X_E \sqrt{Z_\pi} \frac{e^{-E_\pi t_\pi}}{2E_\pi}, \]

where \( X_E = X_{E^-} + X_{E^+} \) and

\[
X_{E^-} = -\sum_n \frac{\left\langle \pi(p) | J(0) | n \right\rangle \left\langle n | H(0) | K \right\rangle}{E_K - E_n} \left( 1 - e^{(E_K - E_n)T_a} \right) \quad \text{and}
\]

\[
X_{E^+} = \sum_{n_s} \frac{\left\langle \pi(p) | H(0) | n_s \right\rangle \left\langle n_s | J(0) | K \right\rangle}{E_{n_s} - E_\pi} \left( 1 - e^{-(E_{n_s} - E_\pi)T_b} \right).
\]

In practice we may need to modify the above formulae to recognise the discrete nature of the lattice.

For \( E_K > E_n \) there are unphysical exponentially growing terms which need to be subtracted! This is a common feature in calculations of long-distance effects in Euclidean space. This requires the consideration of \( \pi, \pi\pi \) and \( \pi\pi\pi \) intermediate states.
The generic non-local matrix elements which we wish to evaluate is

\[ X \equiv \int_{-\infty}^{\infty} dt_x d^3x \langle \pi(p) | T[J(0)H(x)] | K(k) \rangle \]

\[ = i \sum_n \left( \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K(k) \rangle}{E_K - E_n + i\epsilon} \right) \]
\[ - i \sum_{n_s} \left( \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K(k) \rangle}{E_{n_s} - E_\pi + i\epsilon} \right), \]

\{ |n\rangle \} and \{ |n_s\rangle \} represent complete sets of non-strange and strange states.

In Euclidean space we calculate correlation functions of the form

\[ C \equiv \int_{-T_a}^{T_b} dt_x \int d^3x \langle \phi_\pi(\vec{p}, t) \rangle T[J(0)H(x)] \phi_K(\vec{p}_K, t_K) \rangle \equiv \sqrt{Z_K} \frac{e^{-E_K t_K}}{2m_K} X_E \sqrt{Z_\pi} \frac{e^{-E_\pi t_\pi}}{2E_\pi}, \]

where \( X_E = X_{E_-} + X_{E_+} \) and

\[ X_{E_-} = - \sum_n \left( \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K(k) \rangle}{E_K - E_n} \right) \left( 1 - e^{(E_K - E_n)T_a} \right) \]

\[ X_{E_+} = \sum_{n_s} \left( \frac{\langle \pi(p) | H(0) | n_s \rangle \langle n_s | J(0) | K(k) \rangle}{E_{n_s} - E_\pi} \right) \left( 1 - e^{-(E_{n_s} - E_\pi)T_b} \right). \]
For illustration, I consider the kaon to be at rest.

\[ X_{E_0} = - \sum_n \frac{\langle \pi(p) | J(0) | n \rangle \langle n | H(0) | K \rangle}{E_K - E_n} \left( 1 - e^{(E_K - E_n)T_a} \right) \]

We use two methods to remove the contribution from the single pion state.

1. We determine the matrix elements \( \langle \pi | H | K \rangle \) and \( \langle \pi | J | \pi \rangle \) and the energies from two and three-point correlations functions and then perform the subtraction directly.

2. We add a term \( c_S \bar{s}d \) to the effective Hamiltonian, with \( c_S \) chosen for each momentum so that

\[ \langle \pi | H - c_S \bar{s}d | K \rangle = 0. \]

The theoretical demonstration that the addition of a term proportional to \( \bar{s}d \) does not change the physical amplitude can be found in our paper arXiv:1507.03094.
In the continuum, space-time symmetries protect us from two-pion intermediate states:

\[ \langle \pi(p_1) | J_\mu | \pi(p_2) \pi(p_3) \rangle = \epsilon_{\mu \nu \rho \sigma} p_1^\nu p_2^\rho p_3^\sigma F(s, t, u) \]

After integrating over the momenta of the two intermediate pions, the only independent vectors are \(k, p\) and \(\epsilon_\gamma\) and so the indices of the Levi-Civita tensor cannot be saturated.

This still leaves lattice artefacts two-pion contributions \((\propto a^2)\) amplified by the growing exponential factors. While we expect these to be very small (as is the case for \(\Delta m_K\)), this will have to be confirmed numerically.

Recently we have also determined the finite-volume corrections for the two-pion contribution to \(\Delta m_K = m_{KL} - m_{KS}\). N.H.Christ, X.Feng, G.Martinelli & CTS, arXiv:1504.01170

The three pion contribution

- The finite-volume effects which vanish as powers of the volume are absent from diagram (a) for $q^2 < 4m^2$.
- The three-pion on-shell intermediate state contribution is heavily phase-space suppressed and is expected to be negligible (but in principle is also calculable as with method 1 for the single pion contribution).
- The suppression of finite-volume effects which only vanish as powers of the volume due to 2 or 3 particle on-shell intermediate states follows in a similar way.
- (It is only recently that the finite-volume corrections for three particle states have become understood theoretically, but the theory has not been applied in numerical calculations.)

Short Distance Effects

\[ T^\mu_i = \int d^4 x e^{-i q \cdot x} \langle \pi(p) | T \{ J^\mu(x) Q_i(0) \} | K(k) \rangle, \]

- Each of the two local \( Q_i \) operators can be normalized in the standard way and for \( J \) we imagine taking the conserved vector current.
- We must treat additional divergences as \( x \to 0 \).

Quadratic divergence is absent by gauge invariance \( \Rightarrow \) Logarithmic divergence.

- Checked explicitly for Wilson and Clover at one-loop order.
  
  G.Isidori, G.Martinelli and P.Turchetti, hep-lat/0506026

- Absence of power divergences does not require GIM.
- Logarithmic divergence cancelled by GIM.
In the calculation described below we have followed the IMT approach, using the conserved vector current with DWF.

This is not possible for $K \rightarrow \pi \nu \bar{\nu}$ decays because the axial current is present so that the GIM mechanism does not result in the absence of logarithmic divergences. This is discussed in some detail below.

I will illustrate the procedure for handling such additional short-distance divergences by considering $K \rightarrow \pi \ell^+ \ell^-$ decays in the 3-flavour theory, i.e. at scales below the charm-quark mass.

(For $\Delta m_K$ in the 4-flavour theory, the chiral structure of the weak Hamiltonian leads to an absence of any additional divergences. This is not the case for $\varepsilon_K$.)
Many diagrams to evaluate!

- For example for $K^+$ decays we need to evaluate the diagrams obtained by inserting the current at all possible locations in the three point function (and adding the disconnected diagrams):

  \[ K \rightarrow \ell \bar{s} \ell \pi \]

  \[ K \rightarrow \ell \bar{s} \ell \pi \]

  \[ K \rightarrow \ell \bar{u}, \bar{c} \ell \pi \]

  \[ K \rightarrow \ell \bar{u}, \bar{c} \ell \pi \]

  \[ W=Wing, \ C=Connected, \ S=Saucer, \ E=Eye. \]

  \[ K_S \] decays there is an additional topology with a gluonic intermediate state.
The numerical study is performed on the $24^3 \times 64$ DWF+Iwasaki RBC-UKQCD ensembles with $am_l = 0.01$ ($m_\pi \simeq 420$ MeV), $am_s = 0.04$ ($m_K \simeq 625$ MeV), $a^{-1} \simeq 1.78$ fm.

128 configurations were used with $\vec{k} = \vec{0}$ and $\vec{p} = (1,0,0), (1,1,0)$ and $(1,1,1)$ in units of $2\pi/L$.

With this kinematics we are in the unphysical region, $q^2 < 0$.

The charm quark is also lighter than physical $m_c^{\overline{MS}}(2\text{GeV}) \simeq 520$ MeV.

The calculation is performed using the (5-dimensional) conserved vector current.

Disconnected diagrams not included.
Method 1 for $\vec{p} = (1, 0, 0)$

$$\int_{t_J - T_A}^{t_J + 8} \tilde{\Gamma}_0^{(4)} dt_H$$

$$\int_{t_J - 6}^{t_J + T_B} \tilde{\Gamma}_0^{(4)} dt_H$$

$$A_0(q^2) = -0.0028(6).$$
Method 2 for $\vec{p} = (1, 0, 0)$

$$A_0(q^2) = -0.0027(6).$$
Numerical check that the matrix element with $H$ replaced by $\bar{s}d$ is consistent with zero.

$$A_0^{\bar{s}d}(q^2) = -0.00007(8).$$
Summary and Prospects for computations of $K \rightarrow \pi \ell^+ \ell^-$ Decays

For $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ or $K_S \rightarrow \pi^0 \ell^+ \ell^-$ decays we now have a “complete” theoretical framework with which to perform lattice computations of the amplitudes.

- Results from exploratory numerical simulations are very encouraging.
- To use this framework in a simulation with physical quark masses requires a major project.
- This will undoubtedly happen on the timescale of 2-3 years. Additional motivation is provided by the promise from NA62 that new more precise data will be available.

We have seen the practical advantages of working with a propagating charm quark (no additional UV divergences, no penguin operators) as well as the absence of perturbation theory at scales $\mu \leq m_c$.

- For a computation at physical masses we need to have a large enough volume to accommodate propagating pions and simultaneously a sufficiently fine lattice to accommodate the charm quark.
- As an intermediate step, it makes sense to perform a computation integrating out the charm quark. I will use this as an example of the handling of the short-distance divergence which is unavoidable for $K \rightarrow \pi \nu \bar{\nu}$ decays (and for $\epsilon_K$).
The short-distance divergence

We now consider the calculation of the $K \to \pi \ell^+ \ell^-$ decay amplitudes in the 3-flavour theory with effective Hamiltonian:

$$H_W^{(3)} \simeq \frac{G_F}{\sqrt{2}} V^*_{us} V_{ud} \sum_{i=1}^{6,7} z_i(\mu) O_i(\mu)$$

- We drop terms proportional to $V_{td} V_{ts}$.
- $O_{1,2}$ are the current-current operators, $O_3 - O_6$ are QCD penguin operators (now generated by charm quark) and
  $$O_{7V} = (\bar{s} \gamma^\mu_L d) (\bar{\ell} \gamma_\mu \ell).$$
- In the $\overline{\text{MS}}$ renormalisation scheme the Wilson coefficients $z_i$ are known.
  

- The local operator contributes directly to the amplitude, whereas the operators $O_1 - O_6$ contribute through the matrix element of the bilocal operators,
  $$\int d^4x \, \langle \pi | T[O_i(x) J^\mu(0)] | K \rangle.$$  

  The two contributions mix under renormalisation.
The local operators $O_1 - O_6$ are renormalised in the standard way:

$$O^L_i \xrightarrow{\text{NPR}} O^\text{RI-SMOM}_i \xrightarrow{\text{pert}} O^\text{MS}_i.$$ 

Define the bilocal operator in the RI-SMOM scheme by

$$O^\text{RI}_B, i(0)_{\mu_{RI}} = \sum \left\{ T \left[ O^\text{RI}_i(\boldsymbol{x})_{\mu_{RI}} J^\mu(0) \right] \right\}_a - X_i(a, \mu_{RI}) O^\text{RI}_0(0)_{\mu_{RI}},$$

where $O_0 = (\Box g^{\mu\nu} - \partial^\mu \partial^\nu) (\bar{s} \gamma^\nu_L d)$.

The $X_i$ can be determined by imposing a condition such as

$$\langle \bar{d} | O^\text{RI}_B, i(0)_{\mu_{RI}} | \bar{s} \rangle \bigg|_{p_i^2=\mu_{RI}^2} = 0.$$
Having defined and determined the $X_i$ we now need to match the $O_{B,i}^{\text{RI}}$ onto the $\overline{\text{MS}}$ scheme used to determine the Wilson coefficients $y_i$. We write

$$O_{B,i}^{\overline{\text{MS}}} (0)_\mu = Z_{ij}^{\text{RI} \rightarrow \overline{\text{MS}}} (\mu, \mu_{RI}) O_{B,j}^{\text{RI}} (0)_{\mu_{RI}} + Y_i (\mu, \mu_{RI}) O_0^{\text{RI}} (0)_{\mu_{RI}}$$

where the $Y_i$ are determined by calculating

$$\langle \bar{d} | O_{B,i}^{\overline{\text{MS}}} (0)_\mu | \bar{s} \rangle \bigg|_{p_i^2 = \mu_{RI}^2} = Y_i (\mu, \mu_{RI}) \langle \bar{d} | O_0^{\text{RI}} (0)_{\mu_{RI}} | \bar{s} \rangle \bigg|_{p_i^2 = \mu_{RI}^2}.$$ 

A similar procedure is used to calculate the long-distance contributions to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ decay amplitudes and to $\varepsilon_K$.

The operator $O_0 = (\Box g^{\mu \nu} - \partial^\mu \partial^\nu) (\bar{s} \gamma^\nu_L d)$ together with the photon propagator and leptonic current $\Rightarrow O_{7V}$. 
The issues encountered in $K^+ \rightarrow \pi^+ \ell^+ \ell^-$ decays (additional ultra-violet divergences, subtraction or suppression of growing unphysical exponential terms and FV effects which fall as powers of the volume) must also be dealt with here.

In particulars the contributions from the axial current and the breaking of chiral symmetry by the mass terms $\Rightarrow$ the logarithmic divergences which are proportional to $m_q^2$ are not cancelled by the GIM mechanism.

There are now more contributions to consider, including ones containing lepton propagators.


The exploratory numerical results are surprisingly (to me) encouraging.
For this doubly weak decay there are a number of novel diagrams to evaluate:

\[ K^+ \rightarrow \pi^+ \nu \bar{\nu} \quad \text{Decays - WW-Diagrams} \]

\[ \mathcal{H}_{\text{eff}}^{\text{LO}} = -i \frac{G_F}{\sqrt{2}} \sum_{q, \ell} \left( V_{qs}^* O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0} \right) - i \frac{G_F}{\sqrt{2}} \sum_q \lambda_q O^W_q - i \frac{G_F}{\sqrt{2}} \sum_\ell O^Z_\ell, \]

\[ O_{q\ell}^{\Delta S=1} = C_{\Delta S=1}^{\overline{\text{MS}}} (\mu) \left[ (\bar{s}q)_{V-A} (\bar{\nu}_\ell \ell)_{V-A} \right]^{\overline{\text{MS}}} (\mu), \]

\[ O_{q\ell}^{\Delta S=0} = C_{\Delta S=0}^{\overline{\text{MS}}} (\mu) \left[ (\bar{\ell} \nu_\ell)_{V-A} (\bar{q}d)_{V-A} \right]^{\overline{\text{MS}}} (\mu). \]
Z-exchange Diagrams

\[
\mathcal{H}_{\text{eff}}^{\text{LO}} = -i \frac{G_F}{\sqrt{2}} \sum_{q, \ell} \left( V_{qs}^{*} O_{q\ell}^{\Delta S=1} + V_{qd} O_{q\ell}^{\Delta S=0} \right) - i \frac{G_F}{\sqrt{2}} \sum_q \lambda_q O_{q}^W - i \frac{G_F}{\sqrt{2}} \sum_{\ell} O_{\ell}^Z,
\]

\[
O_{q\ell}^W = C_{1q}^{\overline{\text{MS}}} (\mu) Q_{1,q}^{\overline{\text{MS}}} (\mu) + C_{2q}^{\overline{\text{MS}}} (\mu) Q_{2,q}^{\overline{\text{MS}}} (\mu),
\]

\[
O_{\ell\ell}^Z = C_{Z}^{\overline{\text{MS}}} (\mu) \left[ J_{\mu}^{Z} \bar{\nu}_\ell \gamma^{\mu} (1 - \gamma_5) \nu_\ell \right]^{\overline{\text{MS}}} (\mu)
\]
Details of simulation: 800 configs on a $16^3 \times 32$ lattice with $N_f = 2 + 1$ DWF, $a^{-1} \simeq 1.73$ GeV, $m_\pi \simeq 420$ MeV, $m_K \simeq 563$ MeV and $m_c^{\text{MS}}(2\text{GeV}) \simeq 863$ MeV.

For this unphysical kinematics, we find

$$P_{c,u} = 0.2529(\pm 13)(\pm 32)(-45)$$

and

$$\Delta P_{c,u} = 0.0040(\pm 13)(\pm 32)(-45).$$

Large cancellation between WW and Z-exchange contributions.
5.3 Isospin-breaking effects

This section is based on the following two papers:


Isospin breaking effects

- "Standard" QCD calculations have been performed in the isospin limit, i.e. with $m_u = m_d$, so to improve the precision beyond 1% isospin breaking effects (including electromagnetism) need to be included.

- These are

$$O\left(\frac{m_u - m_d}{\Lambda_{\text{QCD}}}\right) \quad \text{and} \quad O(\alpha),$$

i.e. $O(1\%)$ or so.

- Such calculations for the spectrum have been performed for a few years now, with perhaps the most noteworthy result being

$$m_n - m_p = 1.51(16)(23)\,\text{MeV}$$

to be compared to the experimental value of $1.2933322(4)\,\text{MeV}$.

- I stress that including electromagnetic effects, where the photon is massless of course, required considerable theoretical progress, e.g.

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \cdots \Rightarrow \frac{1}{L^3T} \sum_k \frac{1}{k^2} \cdots$$

and we have to control the contribution of the zero mode in the sum.
Calculating electromagnetic corrections to decay amplitudes has an added major complication, not present in computations of the spectrum, the presence of infrared divergences.

This implies that when studying weak decays, such as e.g. $K^+ \rightarrow \ell^+ \nu$ the physical observable must include soft photons in the final state

$$\Gamma(K^+ \rightarrow \ell^+ \nu_\ell (\gamma)) = \Gamma(K^+ \rightarrow \ell^+ \nu_\ell) + \Gamma(K^+ \rightarrow \ell^+ \nu_\ell \gamma).$$

F. Bloch and A. Nordsieck, PR 52 (1937) 54

In 2015 we proposed a method for including electromagnetic corrections in decay amplitudes and are developing it further as well as testing it numerically.


I stress (and will explain) that in order to implement this method successfully, it will be necessary to work with the experimental community to ensure that we are calculating quantities which correspond to the experimental measurements.
Infrared Divergence - Example

\[
I \sim \int_{\text{small } k} d^4 k \frac{1}{(k^2 + i\epsilon)((p_\mu - k)^2 - m_\mu^2 + i\epsilon)((p_\pi - k)^2 - m_\pi^2 + i\epsilon)}
\]

\[
\sim \int_{\text{small } k} d^4 k \frac{1}{k^2(-2p_\mu \cdot k)(-2p_\pi \cdot k)}
\]

\[
\sim \int_{\text{small } k} d^4 k \frac{1}{k^4} \Rightarrow \text{infrared divergence}.
\]

This leads to a contribution to \( \Gamma_0 \) of

\[
\Gamma_0^{\pi\mu} = \Gamma_0^{\text{tree}} \frac{\alpha}{4\pi} \left( \frac{2(1 + r_\mu^2)}{1 - r_\mu^2} \log r_\mu^2 \log \left( \frac{m_\pi^2}{m_\gamma^2} \right) \right) + \cdots
\]

where the photon mass, \( m_\gamma \), is introduced to regulate the infrared divergences and \( r_\mu = m_\mu / m_\pi \).
Infrared Divergence - Example (Cont)

\[ \Gamma_{\pi\mu}^{\text{tree}} = \Gamma_{\pi\mu}^0 = \frac{\alpha}{4\pi} \left( \frac{2(1 + r_\mu^2)}{1 - r_\mu^2} \log r_\mu^2 \log \left( \frac{m_\pi^2}{m_\gamma^2} \right) + \cdots \right), \]

where \( r_\mu = m_\mu/m_\pi \).
Calculating electromagnetic corrections to decay amplitudes has an added major complication, not present in computations of the spectrum, the presence of infrared divergences.

This implies that when studying weak decays, such as e.g. $K^+ \rightarrow \ell^+ \nu$ the physical observable must include soft photons in the final state

$$\Gamma(K^+ \rightarrow \ell^+ \nu_\ell(\gamma)) = \Gamma(K^+ \rightarrow \ell^+ \nu_\ell) + \Gamma(K^+ \rightarrow \ell^+ \nu_\ell\gamma).$$

F. Bloch and A. Nordsieck, PR 52 (1937) 54

The question for the lattice community is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects.

This is a generic problem if em corrections are to be included in the evaluation of a decay process.

In 2015 we proposed a method for including electromagnetic corrections in decay amplitudes and are developing it further as well as testing it numerically.


I stress (and will explain) that in order to implement this method successfully, it will be necessary to work with the experimental community to ensure that we are calculating quantities which correspond to the experimental measurements.
For illustration, I consider leptonic decays of the pion but the discussion is general.

- The discussion applies to the leptonic and semileptonic decays of all pseudoscalar mesons and can be readily adapted to other processes.
- We do not use ChPT. For a ChPT based discussion of $f_\pi$, see J.Gasser & G.R.S.Zarnauskas, arXiv:1008.3479

At $O(\alpha^0)$

$$
\Gamma(\pi^+ \rightarrow \ell^+ \nu_\ell) = \frac{G_F^2 |V_{ud}|^2 f_\pi^2}{8\pi} m_\pi m_\ell^2 \left( 1 - \frac{m_\ell^2}{m_\pi^2} \right)^2 .
$$

- All the hadronic effects are contained in the leptonic decay constant $f_\pi$.

$$
\langle 0 | \bar{d} \gamma^\mu \gamma^5 u | \pi^+ (p) \rangle = ip^\mu f_\pi .
$$
Lattice computations of $\Gamma(\pi^+ \rightarrow \ell^+ \nu_{\ell}(\gamma))$ at $O(\alpha)$

$$
\Gamma(\pi^+ \rightarrow \ell^+ \nu_{\ell}(\gamma)) = \Gamma(\pi^+ \rightarrow \ell^+ \nu_{\ell}) + \Gamma(\pi^+ \rightarrow \ell^+ \nu_{\ell}\gamma)
\equiv \Gamma_0 + \Gamma_1
$$

- In principle, particularly as techniques and resources improve in the future, it may be better to compute $\Gamma_1$ nonperturbatively over a larger range of photon energies.
- At present we do not propose to compute $\Gamma_1$ nonperturbatively. Rather we consider only photons which are sufficiently soft for the point-like (pt) approximation to be valid.

- For pions and kaons at least, a cut-off $\Delta E$ of $O(10-20\text{MeV})$ appears to be appropriate both experimentally and theoretically.
  
  F.Ambrosino et al., KLOE collaboration, hep-ex/0509045. arXiv:0907.3594

- Question: What is the best way to translate the photon energy and angular resolutions at LHC, Belle II etc. into the rest frame of the decaying mesons?
We now write

\[ \Gamma_0 + \Gamma_1(\Delta E) = \lim_{V \to \infty} (\Gamma_0 - \Gamma_{0}^{pt}) + \lim_{V \to \infty} (\Gamma_{0}^{pt} + \Gamma_1(\Delta E)). \]

- \( \Gamma_1 \) stands for point-like.
- The second term on the rhs can be calculated in perturbation theory. It is infrared convergent, but does contain a term proportional to \( \log \Delta E \).
- The first term is also free of infrared divergences.
- \( \Gamma_0 \) is calculated non-perturbatively and \( \Gamma_0^{pt} \) in perturbation theory.

Finite-volume effects take the form:

\[ \Gamma_0^{pt}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell) \log(m_\pi L) + \frac{C_1(r_\ell)}{m_\pi L} + \ldots, \]

where \( r_\ell = m_\ell/m_\pi \) and \( m_\ell \) is the mass of the final-state charged lepton.

The exhibited terms are universal, i.e. independent of the structure of the meson!
- We have calculated the coefficients (using the \( \text{QED}_L \) regulator of the zero mode).
- The leading structure-dependent FV effects in \( \Gamma_0 - \Gamma_0^{pt} \) are of \( O(1/L^2) \).

\[ \Gamma_0 + \Gamma_1(\Delta E) = \lim_{V \to \infty} (\Gamma_0 - \Gamma_0^{pt}) + \lim_{V \to \infty} (\Gamma_0^{pt} + \Gamma_1(\Delta E)). \]

1. Introduction
2. What is $G_F$ at $O(\alpha)$?
3. Proposed calculation of $\Gamma_0 - \Gamma_0^{pt}$
4. Calculation of $\Gamma_0^{pt} + \Gamma_1(\Delta E)$
5. Estimates of structure dependent contributions to $\Gamma_1(\Delta E)$
6. Summary and Conclusions

Here instead I will focus on the finite-volume corrections to $\Gamma_0 - \Gamma_0^{pt}$.
Universal FV effects

Writing

\[ \Gamma_0^{pt}(L) = \Gamma_0^{tree} \left\{ 1 + 2 \frac{\alpha}{4\pi} Y(L) \right\}, \]

we find

\[
Y(L) = \left(1 + r_\ell^2\right) \left[ 2(K_{31} + K_{32}) + \frac{\left(\gamma_E + \log \left[ \frac{L^2 m_\pi^2}{4\pi} \right] \right) \log [r_\ell^2]}{(1 - r_\ell^2)} + \frac{\log^2 [r_\ell^2]}{2(1 - r_\ell^2)} \right] + \\
+ \frac{(1 - 3 r_\ell^2) \log [r_\ell^2]}{(1 - r_\ell^2)} - \log \left[ \frac{M_W^2}{m_\pi^2} \right] + \log[m_\pi^2 L^2] - \frac{1}{2} K_P + \frac{1}{12} + \\
+ \frac{1}{m_\pi L} \left( \frac{2r_\ell^2}{1 - r_\ell^2} \left( K_{21} + K_{22} - 2\pi \left( \frac{1}{1 + r_\ell^2} + \frac{1}{r_\ell^2} \right) \right) - \frac{\pi(1 + r_\ell^2)}{(1 - r_\ell^2)} (K_{11} + K_{12} - 3) \right),
\]

where \( r_\ell = m_\ell / m_\pi \) and the \( K_{ij} \) are constants (\( K_{21} + K_{22} \) and \( K_{31} + K_{32} \) depends on the direction of \( \vec{p}_\ell \)).
Computing the Finite-Volume Effects

- Let $f(p^2)$ be a smooth function. For a sufficiently large $L$:
  $$\frac{1}{L} \sum_n f(p^2_n) = \int \frac{dp}{2\pi} f(p^2),$$
  where $p_n = (2\pi/L)n$ and the relation holds "locally".

- In actual lattice calculations the spacing between momenta are $O(\text{few 100 MeV})$ so we would not expect such a local relation to be sufficiently accurate.

- However using the Poisson summation formula:
  $$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{n=-\infty}^{\infty} \exp(2\pi i nx)$$
  we obtain the powerful exact relation
  $$\frac{1}{L} \sum_{n=-\infty}^{\infty} f(p^2_n) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2) + \sum_{n\neq 0} \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2)e^{inpL},$$
  which for standard calculations implies that
  $$\frac{1}{L} \sum_n f(p^2_n) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} f(p^2),$$
  up to exponentially small corrections in $L$.

- In our approach, this is the starting point for all calculations of FV effects.

Chris Sachrajda
Hsinchu, 18th May 2017
Consider first a standard one-dimensional example:

\[
\int_{-\infty}^{\infty} \frac{dp}{(2\pi)} \frac{e^{ipL}}{p^2 + m^2} = \int_{-\infty}^{\infty} \frac{dp}{(2\pi)} \frac{e^{ipL}}{(p + im)(p - im)} = \frac{1}{m} e^{-mL}.
\]

This is the usual case in FV simulations. The FV effects are exponentially suppressed in the volume.

The photon however, is massless which means:

- The zero-mode must be handled. By now the standard technique is to use QED, i.e. to set \( A_\mu(k_0, \vec{k} = 0) = 0 \).
- All terms on the rhs of the Poisson summation formula need to be summed.

For the spectrum

\[
m_\pi(L) = m_\pi \left[ 1 - q^2 \alpha \left( \frac{\kappa}{m_\pi L} \left( 1 + \frac{2}{m_\pi L} \right) \right) + O \left( \frac{1}{(m_\pi L)^3} \right) \right]
\]

where \( \kappa = 1.41865 \) is a universal constant and the structure dependent terms start at \( O(1/L^3) \).

Our goal was to extend these techniques to decay amplitudes which include infrared divergences.
Our main result was presented on slide 75.

We found a nice scaling law for a function which behaves as $1/(k^2)^{\frac{n}{2}}$ as $k \to 0$:

$$
\xi' = \int \frac{dk_0}{2\pi} \left( \frac{1}{L^3} \sum_{k \neq 0} - \int \frac{d^3 k}{(2\pi)^3} \right) \frac{1}{(k^2)^{\frac{n}{2}}} = O\left( \frac{1}{L^{4-n}} \right)
$$

For the spectrum $n = 3$ and the FV corrections are $O(1/L)$.

For decay amplitudes $n = 4$ and we have the form:

$$
\Gamma_{0}^{\text{pt}}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell) \log (m_\pi L) + \frac{C_1(r_\ell)}{m_\pi L} + \ldots,
$$

where $r_\ell = m_\ell/m_\pi$ and $m_\ell$ is the mass of the final-state charged lepton.

The exhibited terms are universal, i.e. independent of the structure of the meson! They therefore cancel out of the difference $\Gamma_0 - \Gamma_{0}^{\text{pt}}$. 
For "standard" quantities such as $f_K/f_\pi$, $B_K$ or $f^+(0)$, the precision of lattice calculations is now $O(1\%)$ or better → isospin breaking effects, including electromagnetic corrections, must be included.  

FLAG collaboration, arXiv:1607.00299

- A number of groups are studying the IB effects in the spectrum.
- We have proposed a strategy for computing electromagnetic corrections to decay amplitudes and calculated the universal FV corrections up to and including $O(1/L)$. (At $O(1/L^2)$ the corrections are structure dependent.)
In these lectures I have tried to stress

- the importance of flavour physics in exploring the limits of the Standard Model and in searching for signatures of new physics;
- the rôle that lattice QCD is playing, and will continue to play, in evaluating the hadronic effects necessary for precision flavour physics.